Green Functions of Relativistic Field Equations

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Abstract

In this paper, we restudy the Green function expressions of field equations. We derive the explicit form of the Green functions for the Klein-Gordon equation and Dirac equation, and then estimate the decay rate of the solution to the linear equations. The main motivation of this paper is to show that: (1). The formal solutions of field equations expressed by Green function can be elevated as a postulate for unified field theory. (2). The inescapable decay of the solution of linear equations implies that the whole theory of the matter world should include nonlinear interaction.

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1 Introduction

Since the Green function or propagator of the wave equation was introduced to calculate the scattering matrix by Stückelberg[1] and Feynman[2], it has become an important tool of quantum field theory[3, 4]. Green function is also a useful tool for asymptotic analysis and L^∞ estimation of fields. The Feynman propagators are in the form of Fourier integrals with the fields being approximately treated as plane wave. In some old textbooks[5, 6], the explicit Green functions of the wave equations were expressed by Bessel function. In this pedagogical paper, we restudy this problem in detail. We derive the Green functions of the Klein-Gordon equation and Dirac equation as normal

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functions, and express the general solutions of the field equations in integral form, then use this form to estimate the decay rate of the linear fields.

The main motivation of this paper is to show that, (1). the solutions of field equation expressed by Green function are more naturally and visually display the correlation of fields than differential equations, and the expression of integral form are more flexible and comprehensive. This feature of the expression enables us to elevate it as a postulate for unified field theory. (2). The estimation of decay rate of linear fields shows that, the linear field theory is not complete enough to describe the stability of the fundamental particles, A whole theory of the matter world should include nonlinear interaction.

2 Preparation

In this paper we only consider the distribution defined in $R_+ = [0, +\infty)$. By straightforward calculation, we have

Lemma 1. *For $p \in R_+$, in the sense of distribution we have*

$$\frac{2}{\pi}\int_{R_+} \frac{\sin px}{x} dx = H(p), \qquad \frac{2}{\pi}\int_{R_+} \cos px dx = \delta(p), \tag{2.1}$$

where $H(p)$ is the Heaviside function, and $\delta(p)$ is the Dirac-δ

$$H(p) = \begin{cases} 1, & p > 0 \\ 0, & p = 0, \end{cases} \qquad \delta(p) = H'(p). \tag{2.2}$$

By Laplace transformation, we can check

Proof. Denote $\widetilde{f}(p) = \int_{R^+} \cos(px) dx$, for any $f(p) \in C_c^\infty(R_+)$, we have

$$\begin{aligned} \langle f, \widetilde{f} \rangle &\equiv \textstyle\int_{R^+} f(p) \cos(px) dx dp \\ &= \textstyle\lim_{T\to\infty} \int_{R^+} f(p) dp \int_0^T \cos(px) dx = \lim_{T\to\infty} \int_{R^+} f(p) \frac{\sin(pT)}{p} dp \\ &= \textstyle\lim_{T\to\infty} \int_{R^+} f(\frac{\tau}{T}) \frac{\sin(\tau)}{\tau} d\tau = \frac{\pi}{2} f(0). \end{aligned}$$

So we get $\frac{2}{\pi}\int_{R_+} \cos px dx = \delta(p)$. Similarly we can check the other equations.

Lemma 2. *In the sense of distribution, we have*

$$I(p) \equiv \frac{2}{\pi}\int_1^\infty \frac{\sin p\tau}{\sqrt{\tau^2 - 1}} d\tau = J_0(p) H(p), \tag{2.3}$$

where $J_n(p)$ $(n = 0, 1, 2, \cdots)$ is the Bessel function

$$J_n(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(p \sin\theta - n\theta) d\theta = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{p}{2}\right)^{2k+n}. \tag{2.4}$$

Lemma 3. *Suppose $m > 0$. Let $w = \sqrt{m^2 + p^2}$ and*

$$K_1(t, r) = \frac{1}{r}\int_0^\infty \frac{p}{w} \sin wt \sin pr dp, \tag{2.5}$$

then in the sense of distribution, we have

$$K_1(t,r) = \frac{\pi m^2}{2\rho}[J_0(\rho)\delta(\rho) - J_1(\rho)H(\rho)], \quad \rho \equiv m\sqrt{t^2-r^2} \tag{2.6}$$

Proof. Let

$$\begin{cases} w = m\cosh z, \quad p = m\sinh z, \\ t = \frac{\rho}{m}\cosh\xi, \quad r = \frac{\rho}{m}\sinh\xi, \end{cases} \tag{2.7}$$

then we have

$$\begin{aligned} K_1 &= \tfrac{m}{2\rho\sinh\xi}\textstyle\int_{-\infty}^{\infty} \mathrm{th} z\cdot\frac{1}{2}[\cos(wt-pr)-\cos(wt+pr)]dp \\ &= \tfrac{m^2}{4\rho\sinh\xi}\textstyle\int_{-\infty}^{\infty}\sinh z[\cos(\rho\cosh(z-\xi))-\cos(\rho\cosh(z+\xi))]dz \\ &= \tfrac{m^2}{4\rho\sinh\xi}\textstyle\int_{-\infty}^{\infty}\cos(\rho\cosh s)[\sinh(s+\xi)-\sinh(s-\xi)]ds \\ &= \tfrac{m^2}{\rho}\textstyle\int_0^{\infty}\cos(\rho\cosh s)\cosh s ds \\ &= \tfrac{m^2}{\rho}\textstyle\int_1^{\infty}\cos(\rho\tau)\frac{\tau}{\sqrt{\tau^2-1}}d\tau = \frac{\pi m^2}{2\rho}I'(\rho), \end{aligned} \tag{2.8}$$

where I is defined by (2.3). By $J_0' = -J_1$, we prove the lemma.

The following lemmas can be checked directly.

Lemma 4. *For all real vector* $\vec{p} = (p_1, p_2, p_3)$, *denote* $p = \sqrt{p_1^2+p_2^2+p_3^2}$ *and*

$$B_1(\vec{p}) = \frac{1}{\sqrt{2p}}\begin{pmatrix} \frac{p_1-ip_2}{\sqrt{p-p_3}} & -\frac{p_1+ip_2}{\sqrt{p+p_3}} \\ \sqrt{p-p_3} & \sqrt{p+p_3} \end{pmatrix}, \tag{2.9}$$

then B_1 *is a unitary matrix satisfiing*

$$B_1^{-1} = B_1^+, \quad B_1^{-1}\vec{\sigma}\cdot\vec{p}B_1 = p\sigma_3, \tag{2.10}$$

where $\vec{\sigma}$ *are Pauli's matrices.*

Lemma 5. B_1 *is defined by (2.9), denote*

$$\Theta = m\gamma + \vec{\alpha}\cdot\vec{p}, \quad B_2 = diag(B_1, B_1), \tag{2.11}$$

$$\gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \tag{2.12}$$

then we have

$$B_2^{-1}\Theta B_2 = m\gamma + p\alpha_3. \tag{2.13}$$

Lemma 6. *Let* $w = \sqrt{m^2+p^2}$,

$$B_3 = \frac{1}{\sqrt{2w}}\begin{pmatrix} \frac{p}{\sqrt{w-m}} & 0 & \frac{-p}{\sqrt{w+m}} & 0 \\ 0 & \frac{-p}{\sqrt{w-m}} & 0 & \frac{p}{\sqrt{w+m}} \\ \sqrt{w-m} & 0 & \sqrt{w+m} & 0 \\ 0 & \sqrt{w-m} & 0 & \sqrt{w+m} \end{pmatrix}, \tag{2.14}$$

then we have

$$B_3^{-1}(m\gamma + p\alpha_3)B_3 = w\gamma. \tag{2.15}$$

Lemma 7. *Let* $Q = B_2 B_3$, *then*

$$\Theta = wQ\gamma Q^{-1} \quad \text{or} \quad Q^{-1}\Theta Q = w\gamma, \tag{2.16}$$

where all parameters are defined as above.

3 The Green functions of field equations

(F1). For the initial value problem of the Klein-Gordon equation

$$\partial_t^2 u = \Delta u - m^2 u, \quad \text{with} \quad u(0,\vec{x}) \quad \text{and} \quad u_t(0,\vec{x}), \tag{3.1}$$

the formal solution represented by the Green function is given by

$$u(t,\vec{x}) = \int_{R^3} [K(t,\vec{r})u(0,\vec{y}) + G(t,\vec{r})u_t(0,\vec{y})]d^3y, \tag{3.2}$$

where $\vec{r} = \vec{x} - \vec{y}$.

Lemma 8. *In the sense of distribution, for (3.2) we have*

$$G(t,\vec{r}) = \frac{1}{2\pi^2 r}\int_0^\infty \frac{p}{w}\sin wt \sin pr dp, \tag{3.3}$$

$$K(t,\vec{r}) = \partial_t G = \frac{1}{2\pi^2 r}\int_0^\infty p\cos wt \sin pr dp. \tag{3.4}$$

Proof. Denote $u(0,\vec{x}) = \phi(\vec{x})$, $u_t(0,\vec{x}) = \psi(\vec{x})$, Make Fourier transformation to (3.1), we get

$$\begin{cases} \frac{d^2}{dt^2}\widetilde{u} = -(p^2+m^2)\widetilde{u} = -w^2\widetilde{u}, \\ \widetilde{u}(0,\vec{p}) = \widetilde{\phi}(\vec{p}), \quad \widetilde{u}'(0,\vec{p}) = \widetilde{\psi}(\vec{p}), \end{cases} \tag{3.5}$$

where

$$\begin{cases} \widetilde{u}(t,\vec{p}) = \frac{1}{\sqrt{(2\pi)^3}}\int_{R^3} u(t,\vec{y})e^{-\vec{y}\cdot\vec{p}i}d^3y, \\ u(t,\vec{x}) = \frac{1}{\sqrt{(2\pi)^3}}\int_{R^3} \widetilde{u}(t,\vec{p})e^{\vec{x}\cdot\vec{p}i}d^3p, \end{cases} \tag{3.6}$$

and so on. The solution of (3.5) is given by

$$\widetilde{u} = \cos wt\widetilde{\phi} + \frac{1}{w}\sin wt\widetilde{\psi}. \tag{3.7}$$

Substituting (3.7) into (3.6), we get

$$G(t,\vec{r}) = \frac{1}{(2\pi)^3}\int_{R^3}\frac{1}{w}\sin wt e^{(\vec{x}-\vec{y})\cdot\vec{p}i}d^3p, \tag{3.8}$$

$$K(t,\vec{r}) = \frac{1}{(2\pi)^3}\int_{R^3} \cos wt e^{(\vec{x}-\vec{y})\cdot\vec{p}i} d^3p = \partial_t G, \tag{3.9}$$

Let $\vec{r}\cdot\vec{p} = pr\cos\theta$, integrating (3.8) and (3.9) in the spherical coordinate system proves the lemma.

By Lemma 3 and (3.3), we get

Lemma 9. *Let $\rho = m\sqrt{t^2-r^2}$, then we have*

$$G(t,\vec{r}) = \frac{m}{4\pi\sqrt{t^2-r^2}}[J_0(\rho)\delta(\rho) - J_1(\rho)H(\rho)], \tag{3.10}$$

By Lemma 9, we have

Theorem 1. *The solution of (3.1) can be expressed by the following integrals*

$$u(t,\vec{x}) = u_0(t,\vec{x}) - u_1(t,\vec{x}), \tag{3.11}$$

where u_0 is the solution for the case $m = 0$,

$$u_0 = \partial_t\left(\frac{1}{4\pi t}\int_{r=t}\phi(\vec{y})dS_y\right) + \frac{1}{4\pi t}\int_{r=t}\psi(\vec{y})dS_y, \tag{3.12}$$

S_y stands for the surface $r = t$,

$$u_1 = \partial_t\left(\frac{m}{4\pi}\int_{r<t}\frac{J_1(\rho)}{\sqrt{t^2-r^2}}\phi(\vec{y})d^3y\right) + \frac{m}{4\pi}\int_{r<t}\frac{J_1(\rho)}{\sqrt{t^2-r^2}}\psi(\vec{y})d^3y. \tag{3.13}$$

Proof. For simple, we set $\phi = 0$. Then by Lemma 8 and 9, we have

$$\begin{array}{rcl} u(t,\vec{x}) & = & \frac{1}{4\pi}\int_{r<t}\frac{m}{\sqrt{t^2-r^2}}[J_0\delta(\rho) - J_1H(\rho)]\psi(\vec{y})d^3y \\ & = & \frac{1}{4\pi}\int_0^t\frac{m}{\sqrt{t^2-r^2}}J_0\delta(\rho)r^2dr\int_0^\pi\sin\theta d\theta\int_0^{2\pi}\psi d\varphi - u_1(t,\vec{x}) \\ & = & \frac{1}{4\pi}\int_0^t rd(-m\sqrt{t^2-r^2})J_0\delta(\rho)\int_0^\pi\sin\theta d\theta\int_0^{2\pi}\psi d\varphi - u_1(t,\vec{x}) \\ & = & \frac{t}{4\pi}\int_0^\pi\sin\theta d\theta\int_0^{2\pi}\psi(r=t)d\varphi - u_1(t,\vec{x}). \end{array} \tag{3.14}$$

The proof is finished.

(F2). For the Dirac equation

$$\partial_t\psi = -\vec{\alpha}\cdot\nabla\psi - m\gamma i\psi, \quad \text{with} \quad \psi(0,\vec{x}) = \phi(\vec{x}), \tag{3.15}$$

by Fourier transformation we get

Lemma 10. *The formal solution of (3.15) is given by*

$$\psi(t,\vec{x}) = \int_{R^3} A(t,\vec{x}-\vec{y})\phi(\vec{y})d^3y, \tag{3.16}$$

where

$$A(t,\vec{r}) = -\frac{im\gamma}{2\pi^2}K_1 + \frac{1}{2\pi^2}\partial_t K_1 + \frac{1}{2\pi^2}\vec{\alpha}\cdot\nabla K_1, \tag{3.17}$$

and $\nabla = (\partial_{y_1}, \partial_{y_2}, \partial_{y_3})$*,* $K_1(t,r)$ *is defined by (2.5) or (2.6).*

Proof. By Fourier transformation we will get

$$A(t,\vec{r}) = \frac{1}{8\pi^3}\int_{R^3} e^{-(\Theta t - \vec{r}\cdot\vec{p})i} d^3p, \tag{3.18}$$

where $\Theta = m\gamma + \vec{\alpha}\cdot\vec{p}$. By Lemma 5 and 7, we have

$$\begin{aligned} e^{-\Theta ti} &= \exp(-iwtQ\gamma Q^{-1}) = Qe^{-w\gamma ti}Q^{-1} \\ &= \cos wt - i\sin wtQ\gamma Q^{-1} \\ &= \cos wt - i\sin wt(m\gamma + \vec{\alpha}\cdot\vec{p}), \end{aligned} \tag{3.19}$$

Substituting it into (3.18) we have

$$A(t,\vec{r}) = \frac{1}{8\pi^3}\int_{R^3} [\cos wt - i\sin wt(m\gamma + \vec{\alpha}\cdot\vec{p})]e^{\vec{r}\cdot\vec{p}i} d^3p. \tag{3.20}$$

Denote

$$\vec{r} = r(\sin\theta_1\cos\varphi_1, \sin\theta_1\sin\varphi_1, \cos\theta_1), \tag{3.21}$$

$$T = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & \cos\theta_1\sin\varphi_1 & -\sin\theta_1 \\ -\sin\theta_1\sin\varphi_1 & \sin\theta_1\cos\varphi_1 & 0 \\ \sin\theta_1\cos\varphi_1 & \sin\theta_1\sin\varphi_1 & \cos\theta_1 \end{pmatrix}, \tag{3.22}$$

then

$$\vec{r} = r(0,0,1)T. \tag{3.23}$$

Making transformation

$$\vec{p} = \vec{k}T, \quad \vec{k} = k(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \tag{3.24}$$

substituting (3.23) and (3.24) into (3.20) we have

$$\vec{\alpha}\cdot\vec{p} = (\beta_1\cos\varphi + \beta_2\cos\varphi)k\sin\theta + \frac{\vec{\alpha}\cdot\vec{r}}{r}k\cos\theta, \tag{3.25}$$

where $\beta_1,\ \beta_2$ are constant matrices, $\vec{r}\cdot\vec{p} = rk\cos\theta$,

$$A(t,\vec{r}) = \tfrac{1}{4\pi^2}\int_0^\infty k^2 dk \int_0^\pi \sin\theta[\cos wt - \tfrac{i}{w}\sin wt(m\gamma + \tfrac{\vec{\alpha}\cdot\vec{r}}{r}k\cos\theta)]e^{ikr\cos\theta}d\theta. \tag{3.26}$$

Integrating (3.26) with respect to θ gives (3.17).

By Lemma 3 and 10, we have

Theorem 2. *The solution of (3.15) can be expressed by*

$$\psi(t,\vec{x}) = \psi_0(t,\vec{x}) - \psi_1(t,\vec{x}), \tag{3.27}$$

where

$$\psi_0 = \partial_t\left(\frac{1}{4\pi t}\int_{r=t}\phi(\vec{y})dS_y\right) - \frac{1}{4\pi t}\int_{r=t}(m\gamma i + \vec{\alpha}\cdot\nabla)\phi(\vec{y})dS_y, \tag{3.28}$$

$$\psi_1 = \partial_t\left(\frac{m}{4\pi}\int_{r<t}\frac{J_1(\rho)\phi(\vec{y})}{\sqrt{t^2-r^2}}d^3y\right) - \frac{m}{4\pi}\int_{r<t}\frac{J_1(\rho)}{\sqrt{t^2-r^2}}(m\gamma i + \vec{\alpha}\cdot\nabla)\phi(\vec{y})d^3y. \tag{3.29}$$

4 The decay rate of the linear fields

Now we use (3.11) and (3.27) to estimate the decay of the solutions. In [7, 8], for wave equation we have

Theorem 3. *Suppose* $\phi \in W^{n+3,1}(R^3),\ \psi \in W^{n+2,1}(R^3)$, *then the solution of*

$$\begin{cases} (\partial_t^2 - \Delta)u = 0, \\ u(0,\vec{x}) = \phi, \quad \partial_t u(0,\vec{x}) = \psi, \end{cases} \tag{4.1}$$

has the following decay estimate

$$||u(t,\cdot)||_{W^{n,\infty}(R^3)} \le \frac{C}{1+t}(||\phi||_{W^{n+3,1}(R^3)} + ||\psi||_{W^{n+2,1}(R^3)}),\ (n \ge 0), \tag{4.2}$$

$$||\partial_t u(t,\cdot)||_{W^{n-1,\infty}(R^3)} \le \frac{C}{1+t}(||\phi||_{W^{n+3,1}(R^3)} + ||\psi||_{W^{n+2,1}(R^3)}),\ (n \ge 1), \tag{4.3}$$

where C *always stands for a positive constant independent of* $\phi,\ \psi$.

For the Klein-Gordon equation (3.1), we have the following estimation.

Theorem 4. *Suppose* $\phi \in W^{3,1}(R^3),\ \psi \in W^{2,1}(R^3)$, *then the solution of (3.1) has the following decay estimate*

$$|u(t,\vec{x})| \le \frac{C}{\sqrt{1+t}}(||\phi||_{W^{3,1}(R^3)} + ||\psi||_{W^{2,1}(R^3)}). \tag{4.4}$$

Proof. By theorem 3, we have

$$|u_0(t,\vec{x})| \le \frac{C}{1+t}(||\phi||_{W^{3,1}(R^3)} + ||\psi||_{W^{2,1}(R^3)}), \tag{4.5}$$

where u_0 is defined by (3.12). Let

$$v(t,\vec{x}) = \int_{r<t} \frac{J_1(\rho)}{\sqrt{t^2-r^2}}\psi(\vec{y})d^3y. \tag{4.6}$$

Since $|J_1| < 1$, for case $t < 1$, we have

$$\begin{aligned} |v(t,\vec{x})| &= \left|\int_0^t \tfrac{r^2 J_1}{\sqrt{t^2-r^2}}dr \int_\Omega d\Omega \int_r^\infty \partial_s\psi ds\right| \\ &\le C\int_0^1 \tfrac{dr}{\sqrt{1-r^2}} \int_{R^3} |\nabla\psi| d^3y \le C||\nabla\psi||_{L^1(R^3)}, \end{aligned} \tag{4.7}$$

where $d\Omega = \sin\theta d\theta d\varphi$. For $t \ge 1$, we have

$$\begin{aligned} |v(t,\vec{x})| &= \left|\left(\int_0^{t-\frac{1}{2}} + \int_{t-\frac{1}{2}}^t\right)\left(\tfrac{r^2 J_1 dr}{\sqrt{t^2-r^2}} \int_\Omega \psi(\vec{y})d\Omega\right)\right| \\ &\le \tfrac{C}{\sqrt{t^2-(t-\frac{1}{2})^2}} \int_{R^3}|\psi|d^3y + \left|\int_{t-\frac{1}{2}}^t \tfrac{r^2 J_1 dr}{\sqrt{(t+r)(t-r)}} \int_\Omega d\Omega \int_r^\infty \partial_s\psi ds\right| \\ &\le \tfrac{C}{\sqrt{t}}\left(||\psi||_{L^1(R^3)} + \int_{t-\frac{1}{2}}^t \tfrac{dr}{\sqrt{t-r}} \int_{R^3}|\nabla\psi|d^3y\right) \le \tfrac{C}{\sqrt{t}}||\psi||_{W^{1,1}(R^3)}. \end{aligned} \tag{4.8}$$

Combine it with (4.7), we get

$$|v(t,\vec{x})| < \frac{C}{\sqrt{1+t}}||\psi||_{W^{1,1}(R^3)}. \tag{4.9}$$

Similarly, we have

$$\left|\partial_t \int_{r<t} \frac{J_1(\rho)}{\sqrt{t^2-r^2}}\phi(\vec{y})d^3y\right| < \frac{C}{\sqrt{1+t}}||\phi||_{W^{2,1}(R^3)}. \tag{4.10}$$

The theorem holds by (4.5), (4.9) and (4.10).

Considering linear the Dirac equation also satisfies the Klein-Gordon equation (3.1), so we have

Corollary 5. *For Dirac equation (3.15), we have the following estimation*

$$|\psi(t,\vec{x})| < \frac{C}{\sqrt{1+t}}||\phi||_{W^{3,1}(R^3,C^4)}. \tag{4.11}$$

5 Discussion and conclusion

From Theorem 2 we learn that, the solution of a physical field **system** of the first order equation $u(t,\vec{x}) = (u_1, u_2 \cdots u_n)^T$ can be generally expressed by the following functional

$$G(u(x,t)) = \int_{t_0}^{t} d\tau \int_{\Omega(\tau)} (F + \vec{K}\cdot\nabla_\xi u)d\xi, \tag{5.1}$$

where $G(u)$ is an invertible and differentiable function group of $u_k(x,t)$, t_0 is any given time less then t, $\cup_{\tau\in[t_0,t)}\Omega(\tau)$ is the dependent domain of $u(x,t)$,

$$F = F(u(\xi,\tau), v(\xi,\tau), \xi, \tau, x, t, t_0), \quad \vec{K} = \vec{K}(u(\xi,\tau), v(\xi,\tau), \xi, \tau, x, t, t_0), \tag{5.2}$$

all the elements of the group F and the vector matrices $\vec{K}$ are normal functions of u and v. v denotes the interaction of the **environment**. The physical meaning of (5.1) is that, the present state of the system is simply determined by its history and **interaction**. Combined with the principle of limited light velocity, Eq.(5.1) inversely implies the first order field equation

$$A^\mu \partial_\mu u = f(u,v). \tag{5.3}$$

The reason to choose (5.1) instead of (5.3) as postulate to describe the field system is that, (5.1) is more flexible and comprehensive, and its visual physical meaning can be understood by a farm folk[9, 10].

By (4.11) we learn that, the solution of a linear wave equation will inescapably decay to zero. To explain the stability of the fundamental particles, a complete theory should include nonlinear Dirac equations[11, 12].

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